Primitive Near-rings

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Near-rings

Definition: A near-ring is a set N together with two binary operations "$+$" and "$\cdot$" such that $(N,+)$ **is a group (not necessarily abelian)**, $(N,\cdot)$ **is a semigroup** and $\forall n_1,n_2,n_3 \in N:\ (n_1+n_2)\cdot n_3 = n_1\cdot n_3+n_2\cdot n_3$ (**right distributivity law**).

A near-ring $(N,+,\cdot)$ is said to be zero symmetric iff $\forall n \in N : n\cdot 0 = 0$. **All our near-rings will be zero symmetric.**

Natural examples: The zero preserving functions of a group under pointwise addition and function composition.

Definition 1. *Let* $(N,+,*)$ *be a near-ring and* $(\Gamma,+)$ *be a group.* Γ *is called an* N*-group iff there exists a multiplication* $\odot$ *such that:*

(1) $\forall \gamma \in \Gamma \forall n_1,n_2 \in N : (n_1+n_2)\odot\gamma = n_1\odot\gamma + n_2\odot\gamma$

(2) $\forall \gamma \in \Gamma \forall n_1,n_2 \in N : (n_1 * n_2)\odot\gamma = n_1\odot(n_2\odot\gamma)$.

N-groups

Let Γ be an N-group and let S be a normal subgroup of Γ.

S is called an N-ideal of Γ if $\forall n \in N \ \forall \gamma \in \Gamma \ \forall s \in S$: $n(\gamma + s) - n\gamma \in S$.

An N-group Γ is called simple if there do not exist non-trivial N-ideals.

Γ is called strongly monogenic if $N\Gamma \neq \{0\}$ and for all $\gamma \in \Gamma$ either $N\gamma = \Gamma$ or $N\gamma = \{0\}$.

N-groups which are simple and strongly monogenic are called N-groups of type 1.

N-groups which are strongly monogenic and do not contain non-trivial N-subgroups are called of type 2 (type 2 implies type 1).

We define $(0 : \Gamma) := \{n \in N | \forall \gamma \in \Gamma : n\gamma = 0\}$. If $(0 : \Gamma) = \{0\}$ then Γ is called faithful.

Primitive Near-rings

Definition (Jacobson Radical of type 1):

$$J_1(N) := \bigcap_{\Gamma \text{ of type } 1} (0 : \Gamma)$$

Definition: $J_2(N) := \bigcap_{\Gamma \text{ of type } 2} (0 : \Gamma)$

Definition: The N near-ring is 1-primitive (2-primitive) if there exists a faithfulf N-group of type 1 (type 2).

Semisimple Near-rings: $N/J_1(N)$ is called a 1-semisimple near-ring. It is a subdirect product of 1-primitive near-rings (similarly for 2-semisimple near-rings).

In case N has an identity and the descending chain condition on N-subgroups of $(N, +)$, then the concept of being 2-primitive and 1-primitive and also the radicals coincide. In contrast to (finite) rings, primitive near-rings do not necessarily have an identity.

Primitive near-rings with identity can be described very satisfying as so called centralizer near-rings. So to say, these centralizer near-rings are the near-ring counterparts to matrix rings over fields in ring theory. If the near-rings do not have an identity, then other concepts are necessary to describe them efficiently.

Centralizer Near-Rings

Definition 2. *Let* $(\Gamma, +)$ *be a group and* $\varnothing \neq S \subseteq \mathrm{End}(\Gamma, +)$*.* $M_S(\Gamma) := \{f : \Gamma \mapsto \Gamma \mid f(0) = 0$ *and* $\forall s \in S : f \circ s = s \circ f\}$*.* $(M_S(\Gamma), +, \circ)$ *is a near-ring, called a centralizer near-ring.*

Theorem 3. *(1970's) Every zero symmetric near-ring with identity is (isomorphic to) a centralizer near-ring* $M_S(\Gamma)$*, for a suitable group* Γ *and* $S \subseteq \mathrm{End}(\Gamma, +)$*.*

Theorem 4. *(Betsch, 1971) Let* N *be a zero symmetric (2-)primitive near-ring (not a ring) with identity. Then* N *is dense in some centralizer near-ring* $M_G(\Gamma)$*, where* G *is a fixedpointfree automorphism group of the group* Γ*.*

In the finite case density means equality.

Sandwich Centralizer Near-Rings

Definition 5. *Let* $(\Gamma, +)$ *be a group,* $X \subseteq \Gamma$ *a subset of* Γ *containing the zero* 0 *of* $(\Gamma, +)$ *and* $\phi : \Gamma \longrightarrow X$ *a map such that* $\phi(0) = 0$*. Define the following operation* $\circ'$ *on* Γ^X*:* $f \circ' g := f \circ \phi \circ g$ *for* $f, g \in \Gamma^X$*. Then* $(\Gamma^X, +, \circ')$ *is a (sandwich) near-ring, denoted by* $M(X, \Gamma, \phi)$*.*

Combination of the concepts of centralizer near-rings and sandwich near-rings yields a new class of near-rings, which we call *sandwich centralizer near-rings*.

Definition 6. *(Sandwich Centralizer Near-Rings)*

Let $\varnothing \neq S \subseteq \mathrm{End}(\Gamma, +)$ *such that* $\forall s \in S\ \forall \gamma \in \Gamma$: $\phi \circ s(\gamma) = s \circ \phi(\gamma)$ *and such that* $S(X) \subseteq X$*. Then* $M_0(X, \Gamma, \phi, S) := \{f : X \longrightarrow \Gamma \mid f(0) = 0$ *and* $\forall s \in S\, \forall x \in X : f(s(x)) = s(f(x))\}$ *is a zero symmetric subnear-ring of* $M(X, \Gamma, \phi)$*.*

Near-Rings with right identity

Theorem 7. *Let N be a near-ring. Then the following are equivalent:*

(1) N is a zero symmetric near-ring with right identity.

(2) There exists a group $(\Gamma, +)$, a subset X of Γ with $0 \in X$, there exists a non-empty subset $S \subseteq \mathrm{End}(\Gamma, +)$ with $S(X) \subseteq X$, and there exists a function $\phi : \Gamma \longrightarrow X$ with $\phi(0) = 0$, $\phi \mid_X = id$ and $\phi \circ s(\gamma) = s \circ \phi(\gamma)$ for all $s \in S$ and $\gamma \in \Gamma$, such that $N \cong M_0(X, \Gamma, \phi, S)$.

Important classes of near-rings with multiplicative right identity are: planar near-rings, **(finite) primitive near-rings**, (finite) semi-simple near-rings, any (finite) near-ring not entirely consisting of zero-divisors. In general, those near-rings do not have an identity.

Theorem 8. *Let M be a zero symmetric near-ring which is not a ring. Then the following are equivalent:*

(1) M *is* 1*-primitive and has a right identity.*

(2) There exist:

a. a group $(N, +)$ *and a subset* X *of* N *containing zero and* $|X| \geq 2$,

b. $S \leq \mathrm{Aut}(N, +)$, *with* $S(X) \subseteq X$ *and* S *acting without fixed points on* X,

c. a function $\phi : N \longrightarrow X$ *with* $\phi\,|_X = id$, $\phi(0) = 0$ *and* $\phi \circ s = s \circ \phi$ *for all* $s \in S$,

such that M *is isomorphic to a subnear-ring* M_S *of the sandwich centralizer near-ring* $M_0(X, N, \phi, S)$. *Furthermore,*

d. for any natural number k *the following holds:* $\forall x_1, \ldots, x_k \in X \setminus \{0\}$, $S(x_i) \neq S(x_j)$ *for* $i \neq j$ $\forall n_1, \ldots, n_k \in N$, $\exists f \in M_S$, $\forall i \in \{1, \ldots, k\} : f(x_i) = n_i$.

e. $(N, +)$ *contains no non-trivial normal subgroup* $(U, +)$ *with the property that for all* $u \in U$ *and all* $n \in N$, *there exists an* $s \in S$ *such that* $\phi(n+u) = s(\phi(n))$ *and* $s(n_1) - n_1 \in U$ *for all* $n_1 \in N$.

Some comments and open cases

In the finite case $M \cong M_0(X, N, \phi, S)$.

The Theorem of the last slide can be easily adapted to 1-primitive near-rings being also 2-primitive.

For primitive near-rings which do not even have a right identity, there is still no satisfying classification available (there exists concepts similar to sandwich centralizer near-rings).

There also exist so called 0-primitive near-rings, corresponding to near-rings having faithful N-groups of type 0 (a weaker condition as being of type 1). For such near-rings no density-like theorems exist at all at the moment.