Quasi-duo rings

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Properties:

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• If R is right quasi-duo semiprimitive ring, then R is a subdirect product of division rings.

• (Lam, Dugas) When R is a subdirect product of finite number of division rings then R is quasi-duo.

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- Which rings can be represented as subdirect products of division rings, and when is such a subdirect product right quasi-duo?
- Does there exist a right quasi-duo ring which is not left quasi-duo?
- Does there exist a left primitive, right quasi-duo ring that is not a division ring?

Theorem

For a ring R with an endomorphism τ, the following conditions are equivalent:

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3. R *and* $A(\bar{R},\tau)[x;\tau]$ *are right (left) quasi-duo rings;*
4. *The following conditions hold:*
    1. R *is right (left) quasi-duo and* $J(R[x;\tau]) = (J(R) \cap N(R)) + N(R)[x;\tau]x$;
    2. $N(R)$ *is a* τ*-stable ideal of* R*, the factor ring* $R/N(R)$ *is commutative and the endomorphism* τ *induces identity on* $R/N(R)$.

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Suppose that σ is an automorphism of R and either R is one-sided noetherian or σ is of locally finite order. Then the following conditions are equivalent:

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- $R[x;\sigma]$ *is right quasi-duo;*
- $J(R[x;\sigma]) = \mathcal{N}(R)[x;\sigma]$, *where* $\mathcal{N}(R)$ *is the nil radical of* R, $R/\mathcal{N}(R)$ *is commutative and the automorphism of* $R/\mathcal{N}(R)$ *induced by* σ *is equal to* $\mathrm{id}_{R/\mathcal{N}(R)}$.
- $R[x;\sigma]/J(R[x;\sigma])$ *is commutative.*

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- $J(R[x]) = \mathcal{N}(R)[x]$ *and the factor ring* $R/\mathcal{N}(R)$ *is commutative, where* $\mathcal{N}(R)$ *denote the nil radical of* R.

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- there exists a skew polynomial ring $R[x;\sigma]$ which is right quasi-duo whereas $R[x;\sigma^{-1}]$ is not.

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Problem

Give necessary and sufficient condition for an Ore extension $R[x;\tau,\delta]$ to be right quasi-duo.

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Proposition

Let R be a $\mathbb{Z}$-graded ring. Then:

(i) $A(R) = A_l = A_r$

(ii) $A(R) \cap (\bigoplus_{0\neq n\in\mathbb{Z}} R_n) = J(R) \cap (\bigoplus_{0\neq n\in\mathbb{Z}} R_n)$.

Theorem

A $\mathbb{Z}$-graded ring R is right (left) quasi-duo if and only if R_0 is right (left) quasi-duo and $R/A(R)$ is a commutative ring.

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Question

Is $\mathcal{R}_l = \mathcal{R}_r$?

Thanks for your attention