On special and nonspecial radicals

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Definition

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(H. J. Le Roux and G. A. P. Heyman) If ρ is a supernilpotent radical, then so is $\mathcal{L}(\rho^*)$ and $\rho \subseteq \mathcal{L}(\rho^*) \subseteq \rho_\varphi$, where ρ_φ denotes the upper radical determined by the class of all subdirectly irreducible rings with ρ-semisimple hearts. Moreover, $\mathcal{L}(\mathcal{G}^*) = \mathcal{G}_\varphi$, where $\mathcal{G}$ is the Brown-McCoy radical.

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Problem

Is it true that $\mathcal{L}(\rho^*) = \rho_\varphi$ if ρ is replaced by β, $\mathcal{L}$, $\mathcal{N}$ or $\mathcal{J}$, where β, $\mathcal{L}$, $\mathcal{N}$ and $\mathcal{J}$ denote the Baer, the Levitzki, the Koethe and the Jacobson radical, respectively?

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Aim of the talk: To give a negative answer to this question.

Lemma

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Let $A \in \rho^*$ and suppose that the ρ-radical $\rho(A)$ of A is nonzero.
Then $A/\rho(A) \in \rho$ and, since $\rho(A) \in \rho$ and ρ is closed under extensions, it follows that $A \in \rho$.

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Corollary

If ρ is a supernilpotent radical, then for any $A \in \rho^*$, either $A \in \rho$ or A is a prime ring.

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If $A \in \rho$, then we are done. So assume that $A \in \mathcal{S}(\rho)$.
Then, since ρ is a supernilpotent radical, A is a semiprime ring. We will now show that A is, in fact, a prime ring.

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Let I and J be ideals of A and suppose that $IJ = 0$ and $I \neq 0$. We will show that $J = 0$.

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Since $(I \cap J)^2 \subseteq IJ = 0$ and A is a semiprime ring, it follows that $I \cap J = 0$.
But $(I + J)/I$ is an ideal of A/I and $A/I \in \rho$ because I is a nonzero ideal of A and $A \in \rho^*$.

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Thus, since ρ being a supernilpotent radical is hereditary, it follows that $(I + J)/I \in \rho$.

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But $(I + J)/I \simeq J/(I \cap J) \simeq J$ since $I \cap J = 0$. Thus $J \in \rho$.

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But $(I + J)/I \simeq J/(I \cap J) \simeq J$ since $I \cap J = 0$. Thus $J \in \rho$.
On the other hand, since $\mathcal{S}(\rho)$ is hereditary and $J \lhd A \in \mathcal{S}(\rho)$, it follows that $J \in \mathcal{S}(\rho)$. Thus $J \in \rho \cap \mathcal{S}(\rho) = \{0\}$ which implies that $J = 0$. □

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1. *The semigroup ring $A(W(\kappa))$ is a subdirect sum of copies of A.*
2. *$A(W(\kappa))$ is prime essential.*
3. *Every prime homomorphic image $A(W(\kappa))/Q$ of $A(W(\kappa))$ is isomorphic to some prime homomorphic image A/P of A.*

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Theorem

If ρ is a supernilpotent radical whose semisimple class $\mathcal{S}(\rho)$ contains a nonzero nonsimple $*$-ring without minimal ideals, then $\mathcal{L}(\rho^*)$ is a nonspecial radical and consequently $\mathcal{L}(\rho^*) \neq \rho_\varphi$.

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Proof.

Let ρ be a supernilpotent radical and let a nonzero nonsimple $*$-ring A without minimal ideals be in $\mathcal{S}(\rho)$. Then $A \in \rho^* \cap \mathcal{S}(\rho)$.
Let $\kappa > 1$ be a cardinal number greater than the cardinality of A and let $A(W(\kappa))$ be the semigroup ring constructed in Theorem 3. Then, by Theorem 3, $A(W(\kappa))$ is prime essential and $A(W(\kappa))$ is a subdirect sum of copies of A. But, since $A \in \mathcal{S}(\rho)$, it follows that $A(W(\kappa)) \in \mathcal{S}(\rho)$ because $\mathcal{S}(\rho)$ is closed under subdirect sums. So $A(W(\kappa)) \in \mathcal{S}(\rho) \cap \mathcal{E}$.
We will now show that $A(W(\kappa)) \in \mathcal{S}(\mathcal{L}(\rho^*))$.
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Proof.

Now, if $\mathcal{L}(\rho^*)$ were a special radical, then by Theorem 4, $A(W(\kappa))$ would contain a family $\{I_\lambda\}_{\lambda\in\Lambda}$ of ideals I_λ such that $\bigcap\limits_{\lambda\in\Lambda} I_\lambda = 0$ and $A(W(\kappa))/I_\lambda \in \mathcal{S}(\mathcal{L}(\rho^*)) \cap \pi$, where π denotes the class of all prime rings.

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$0 \neq A \in \mathcal{L}(\rho^*) \cap \mathcal{S}(\mathcal{L}(\rho^*)) = \{0\}$ and we have a contradiction.

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Example

(E.Sasiada, A.Sulinski) Let F be a field of characteristic 0 which has an authomorphism S such that no integral power of S is the identity automorphism.

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Corollary

If ρ *is replaced by* β, $\mathcal{L}$, $\mathcal{N}$ *or* $\mathcal{J}$, *then* $\rho \subsetneq \mathcal{L}(\rho^*) \subsetneq \rho_\varphi$

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Proof.

It is well known that β, $\mathcal{L}$, $\mathcal{N}$ and $\mathcal{J}$ are special radicals and $\beta \subseteq \mathcal{L} \subseteq \mathcal{N} \subseteq \mathcal{J}$.

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It is well known that β, $\mathcal{L}$, $\mathcal{N}$ and $\mathcal{J}$ are special radicals and $\beta \subseteq \mathcal{L} \subseteq \mathcal{N} \subseteq \mathcal{J}$. Let T be the ring of Example. Clearly, T is a nonzero nonsimple $*$-ring without minimal ideals.

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If ρ is replaced by β, $\mathcal{L}$, $\mathcal{N}$ or $\mathcal{J}$, then $\rho \subsetneq \mathcal{L}(\rho^*) \subsetneq \rho_\varphi$

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